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TIGHT ROUNDS ON THE RESPONSE OF MULTIVARIABLE SYSTEMS WITH COMP--ETC(U)
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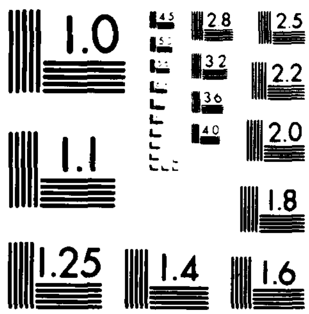

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TIGHT BOUNDS ON THE RESPONSE OF MULTIVARIABLE SYSTEMS WITH COMPONENT UNCERTAINTY*

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ABSTRACT

For Multi-Input-Multi-Output (MIMO) feedback systems having internal components that are subject to parameter variations, nonlinearity, or unmodeled dynamics within given 'conic sector' bounds, tight bounds (which also turn out to take the form of 'conic sector' bounds) are derived for the input-output relation of the closed-loop system. In the case of linear-time invariant systems, the 'conic sector' bounds are interpretable in terms of the maximal singular values of certain frequency-response matrices.

INTRODUCTION

One of the main motivations for the use of feedback in control system design is to reduce the effects of plant ignorance on system response. In the case of single-input-single-output (SISO) feedback systems in which plant ignorance limited to a few (typically, 1 or 2) uncertain parameters known to lie within specified bounds, the Evans root-locus method [1] and Nichols chart frequency-response method [2, p. 197] are very useful (cf. [3, Ch. 6],[4]). In both cases, one graphically constructs 'regions of uncertainty' containing (for all possible parameter values) the corresponding representation of the system response. In the case of root locus approach, these 'regions' are the set of values assumed by each of the system poles as system parameters vary over specified ranges. Similarly, in the Nichol's chart approach the 'regions of uncertainty' are the sets of values assumed by the systems open-loop log magnitude versus phase plot at various frequencies as system parameters vary over specified ranges. Because the 'regions of uncertainty' used in these approaches are difficult to construct for systems with more than one or two uncertain parameters and because these approaches are incompatible with MIMO systems or with systems having nonlinear or time-varying plant ignorance, there is a need for an alternative approach to the problem of bounding the effects of plant uncertainty on closed-loop system response. The results of this paper establish that 'conic sector conditions' provide such an alternative approach. For, linear time-invarisnt systems, these conditions can be expressed in terms of the 'singular values' of certain frequency response matrices.

LEVEL II

NOTATION AND PROBLEM FORMULATION

We consider the class of systems which can be represented as a linear time-invariant (LTI) interconnection of N uncertain multi-input-multi-output components. It is assumed that an 'approximate' LTI model $\underline{C}_i$ ($i = 1,\ldots,N$) and associated transfer function matrix $\underline{C}_i(s)$ is available for each of the N uncertain components, but that the actual system components

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have 'perturbed' input-output relations which may in general include unmodeled 'cross-couplings' between components as well as uncertainty in the input-output relations of the individual components. Thus, we are considering the class of systems having an overall input-output relation

$$\underline{y} = \underset{\sim}{T}\,\underline{u}$$

where the operator $\underset{\sim}{T}$ is defined by feedback equations of the general form (See Figure 1)

$$\begin{bmatrix} \underline{y} \\ \underline{e} \end{bmatrix} = \underset{\sim}{H} \begin{bmatrix} \underline{u} \\ \underline{f} \end{bmatrix} \tag{1a}$$

$$\underline{f} = (\underset{\sim}{C} + \delta\underset{\sim}{C})\,\underline{e} \tag{1b}$$

where

- $\underset{\sim}{H}$ is an LTI operator (having transfer function matrix $\underline{H}(s)$) representing the internal and external interconnections of the system's components;
- $\underset{\sim}{C} = \mathrm{diag}\,(\underset{\sim}{C}_1, \ldots, \underset{\sim}{C}_N)$; (2)[1]
- $\delta\underset{\sim}{C}$ represents the uncertain perturbations and may in general be nonlinear, dynamical, and time-varying;
- $\underline{e} = \mathrm{col}\,(\underline{e}_1, \ldots, \underline{e}_N)$ and

  $\underline{f} = \mathrm{col}\,(\underline{f}_1, \ldots, \underline{f}_N)$;

  $\underline{e}_i$ and $\underline{f}_i$ represent the input and output signals respectively of the i-th 'perturbed' component;
- $\underline{u}$ is the exogenous input to the system;
- $\underline{y}$ is the observed output of the system.

To keep the mathematics tractable, it is assumed that $\underline{H}(s)$ and $\underline{C}(s)$ are rational transfer function matrices -- this is not a serious restriction since most systems of interest with non-rational transfer functions can be approximated arbitrarily closely with respect to the L_2-norm by systems with rational transfer functions.

Our ignorance regarding the accuracy of the 'approximate' models of the system components is assumed to be bounded by an L_2-conic sector. Conic sectors are defined as follows:

<u>Definition</u>: Given any three compatible operators $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$, we define L_2-Cone $(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S})$ to be the set of ordered pairs of signals $(\underline{x},\underline{y}) \in L_2 \times L_2$ satisfying

$$\|\underset{\sim}{S}(\underline{y} - \underset{\sim}{C}\,\underline{s})\|_{L_2} \le \|\underset{\sim}{R}\,\underline{x}\|_{L_2}. \tag{2}$$ [2]

---

[1] For any collection of operators $\underset{\sim}{A}_i$ $(i = 1,\ldots,n)$, the notation $\mathrm{diag}(\underset{\sim}{A}_1,\ldots,\underset{\sim}{A}_n)$ is defined by

$$\mathrm{diag}(\underset{\sim}{A}_1,\ldots,\underset{\sim}{A}_n) \begin{bmatrix} \underline{e}_1 \\ \vdots \\ \underline{e}_n \end{bmatrix} \triangleq \begin{bmatrix} \underset{\sim}{A}_1\underline{e}_1 \\ \vdots \\ \underset{\sim}{A}_n\underline{e}_n \end{bmatrix},$$

for all $\underline{e}_i$ $(i = 1,\ldots,n)$.

[2] We use the notation L_2 to denote the normed space of C^n-valued signals $\underline{z} : R \to C^n$ for some integer n for which the norm $\|\underline{z}\|_{L_2} \triangleq \langle \underline{z}, \underline{z}\rangle_{L_2}$ exists and is finite; $\langle \underline{z}_1, \underline{z}_2\rangle_{L_2} = \int_{-\infty}^{\infty} \underline{z}_1^T(t)\,\underline{z}_2(t)\,dt$ [5]; $\underline{z}_1$ is the complex congugate of $\underline{z}_1$.

radius $\underset{\sim}{R}$, and output radius $\underset{\sim}{S}$. If $\underset{\sim}{H}$ is an operator with the property that $\underset{\sim}{H}-\underset{\sim}{C}$ is L_2-stable[3] and if for all $(\underline{x},\underline{y}) \in L_2 \times L_2$ with $\underline{y} = \underset{\sim}{H}\,\underline{x}$ we have

$$||\underset{\sim}{S}(\underline{y} - \underset{\sim}{C}\,\underline{x})||_{L_2}^2 \leq ||\underset{\sim}{R}\,\underline{x}||_{L_2}^2 \tag{3}$$

then we say

$$\underset{\sim}{H} \ \underline{\text{inside}} \ L_2\text{-Cone}\,(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}). \tag{4}$$[4]

(End of Definition)

It is a trivial consequence of Parseval's theorem (cf. [5, p. 236]) that, in the case of L_2-stable LTI operators $\underset{\sim}{S}$, $\underset{\sim}{R}$, and $\delta\underset{\sim}{C} \triangleq \underset{\sim}{H}-\underset{\sim}{C}$ having respective transfer functions $\underline{S}(s)$, $\underline{R}(s)$ and $\underline{H}(s)-\underline{C}(s)$, the condition (4) is equivalent to the frequency-domain condition

$$\sigma_{MAX}\left(\underline{S}(j\omega)(\underline{H}(j\omega) - \underline{C}(j\omega))\ \underline{R}^{-1}(j\omega)\right) \leq 1 \quad \text{for all } \omega \tag{5}$$[5]

(where $\sigma_{MAX}(\underline{A})$ denotes the largest singular value of the complex matrix $\underline{A}$ i.e., the square root of the largest eigenvalue of A^*A (or equivalently, of $\underline{AA}^*$) -- cf. [7, Ch. 1]). The motivation for the designation of $\underset{\sim}{C}$, $\underset{\sim}{R}$, and $\underset{\sim}{S}$ as the center, input radius, and output radius respectively of L_2-Cone$(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S})$ is that in the special case where $\underset{\sim}{H},\underset{\sim}{C},\underset{\sim}{R}$ and $\underset{\sim}{S}$ are SISO and LTI, then the statement $\underset{\sim}{H}$ inside L_2-Cone $(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S})$ implies that the ω-dependent function $H(j\omega) : \mathbf{C} \to \mathbf{C}$ given by $Y(j\omega) = H(j\omega)\,U(j\omega)$ maps inputs $U(j\omega)$ lying inside the circle of radius $R(j\omega)$ in the complex plane $\mathbf{C}$ into outputs $Y(j\omega)$ lying inside the circle of center $C(j\omega)$ and radius $S(j\omega)$ in the complex plane $\mathbf{C}$.

For memoryless operators $(\underset{\sim}{H}\,\underline{x})(t) = \underline{h}(\underline{x}(t), t)$, $(\underset{\sim}{C}\,\underline{x})(t) = \underline{C}\,\underline{x}(t)$, $(\underset{\sim}{R}\,\underline{x})(t) = \underline{R}\,\underline{x}(t)$, and $(\underset{\sim}{S}\,\underline{y})(t) = \underline{S}\,\underline{y}(t)$, the condition (4) is equivalent to the condition

$$||\underline{S}\left(\underline{h}(\underline{x}, t) = \underline{C}\,\underline{x}\right)||_{R^n} \leq ||\underline{R}\,\underline{x}||_{R^n} \tag{6}$$

(where $||\underline{z}||_{R^n} \triangleq \sqrt{\underline{z}^T\underline{z}}$ for all $\underline{z} \in R^n$). The motivation for using the term 'conic sector' comes from the fact that in the special case where S, $h(\cdot)$, C, and R are scalar, condition (6) implies that for each t the graph of h(x, t) vs. x is in a cone shaped subset of the real plane -- cf. [13].

Our results assume that

$$\underset{\sim}{C} + \delta\underset{\sim}{C} \ \underline{\text{inside}} \ L_2\text{-Cone}(\underset{\sim}{C}, \underset{\sim}{R}_c, \underset{\sim}{S}_c) \tag{7}$$

where $\underset{\sim}{C}$ is as in equation (1) and $(\underset{\sim}{R}_c, \underset{\sim}{S}_c)$ are given. It is further assumed that $(\underset{\sim}{R}_c, \underset{\sim}{S}_c)$ are L_2-stable LTI operators with square, rational transfer function matrices $(\underline{R}_c(s), \underline{S}_c(s))$ having the property that $\underline{R}_c^*(j\omega)\underline{R}_c(j\omega)$ and $\underline{S}_c^*(j\omega)\underline{S}_c(j\omega)$ are uniformly positive definite for all ω.[6]

[3] An operator $\underset{\sim}{H}$ is said to be L_2-stable if $\underset{\sim}{H}\,\underline{x}$ L_2 whenever $\underline{x} \in L_2$ and further, for some constant $k < \infty$, $||\underset{\sim}{H}\,\underline{x}||_{L_2} \leq k\,||\underline{x}||_{L_2}$ [5].

[4] In the multivariable generalization of the circle stability criterion in [6], a set called L_{2e}-Cone$(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S})$ is employed which is closely related, but different, from the set L_2-Cone(C, R, S). In general neither set is a proper subset of the other, but if $\underset{\sim}{H}$, $\underset{\sim}{C}$, $\underset{\sim}{S}$, and $\underset{\sim}{R}^{-1}$ are stable nonanticipative operators, then it can be shown (using the definition in [6] of L_{2e}-Cone$(\cdot,\cdot,\cdot)$) that $\underset{\sim}{H}$ inside L_{2e}-Cone$(\underset{\sim}{C},\underset{\sim}{R},\underset{\sim}{S})$ if and only if $\underset{\sim}{H}$ inside L_2-Cone$(\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S})$.

[5] Of course $\underline{R}^{-1}(j\omega)$ must exist for (5) to be equivalent to (4).

[6] For any matrix $\underline{A}$, the transpose of $\underline{A}$ is denoted $\underline{A}^T$ and the complex congugate of $\underline{A}^T$ is denoted $\underline{A}^*$.

In the special case when there is no ignorance regarding the interconnection structure of the system (i.e., when the perturbation $\delta \underset{\sim}{C}$ contains no 'cross-coupling' between components), then $\delta \underset{\sim}{C}$ takes the special form

$$\delta \underset{\sim}{C} = \mathrm{diag}\,(\delta \underset{\sim}{C}_1, \ldots, \delta \underset{\sim}{C}_N)$$

where $\delta \underset{\sim}{C}_i$ represents the ignorance regarding the dynamics of the i-th component ($i = 1, \ldots, N$). If there are conic sector bounds available for the uncertainty in the individual components' input-output relations

$$\underset{\sim}{C}_i + \delta \underset{\sim}{C}_i \ \underline{\text{inside}}\ L_2\text{-Cone}\,(\underset{\sim}{C}_i, \underset{\sim}{R}_i, \underset{\sim}{S}_i),\ (i = 1, \ldots, N) \tag{8}$$

then it follows (cf. [9, Lemma 6.2 (vi)], [10, Lemma 4.2 (vi)]) that (5) holds with

$$\underset{\sim}{R}_c = \mathrm{diag}\,(\underset{\sim}{R}_1, \ldots, \underset{\sim}{R}_N) \tag{9}$$

$$\underset{\sim}{S}_c = \mathrm{diag}\,(\underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_N) \quad . \tag{10}$$

We denote by $\underset{\sim}{L}$ the LTI operator whose transfer function matrix is (suppressing the argument 's')

$$\underline{L} \equiv \left[\begin{array}{c|c} \underline{L}_{yu} & \underline{L}_{yv} \\ \hline \underline{L}_{eu} & \underline{L}_{ev} \end{array}\right] = \left[\begin{array}{c|c} \underline{H}_{yu} + \underline{H}_{yf}\,\underline{C}(\underline{I} - \underline{H}_{ef}\,\underline{C})^{-1}\underline{H}_{eu} & \underline{H}_{yf}(\underline{I} - \underline{C}\,\underline{H}_{ef})^{-1} \\ \hline (\underline{I} - \underline{H}_{ef}\,\underline{C})^{-1}\,\underline{H}_{eu} & \underline{H}_{ef}(\underline{I} - \underline{C}\,\underline{H}_{ef})^{-1} \end{array}\right] \tag{11}$$

For any full rank rational para-hermitian matrix $\underline{A}(s)$ for which $\underline{A}(j\omega)$ positive definite for all ω, we denote by $A^{1/2}(s)$ the rational spectral factor of $\underline{A}(s)$ (unique to within a constant unitary left multiplier) having the properties that $\underline{A}(s) = (\underline{A}^{1/2}(-s))^T\, \underline{A}^{1/2}(s)$, that $(\underline{A}^{1/2}(s))^{-1}$ exists, and that $\underline{A}^{1/2}(s)$ and its inverse have no poles in $\mathrm{Re}(s) \geq 0$; $\underline{A}^{1/2}(s)$ always exists [8, Theorem 2]. If $\underset{\sim}{A}$ is an operator, not necessarily nonanticipative, whose bilateral Laplace transform transfer function matrix is proper, rational, para-hermitian and has the property that $\underline{A}(j\omega)$ is uniformly positive-definite for all ω, then $\underset{\sim}{A}^{1/2}$ denotes the nonanticipative minimum phase LTI operator having transfer function $\underline{A}^{1/2}(s)$. Given any L_2-stable LTI operator $\underset{\sim}{A}$, we denote by $\underset{\sim}{A}^*$ the L_2 adjoint operator; i.e., denoting by $A_\delta(t)$ the impulse response matrix of $\underset{\sim}{A}$, $\underset{\sim}{A}^*$ is the LTI operator with impulse response matrix $\underline{A}_\delta^T(-t)$. Note that if $\underline{A}(s)$ is the bilateral Laplace transform transfer function matrix of $\underset{\sim}{A}$, then $A^T(-s)$ is the transfer function matrix of $\underset{\sim}{A}^*$ and further $\underline{A}^T(-j\omega) = \underline{A}^*(j\omega)$.

MAIN RESULT

Our main result is the following theorem giving $\underline{\text{tight}}$ bounds on the overall systems input-output relation $\underset{\sim}{T}$.

__Theorem 1__: Suppose that (7) holds. If

(a) uniformly for all ω

$$\sigma_{min}\,(\underline{S}_c(j\omega)\ \underline{L}_{ev}^{-1}(j\omega)\ \underline{R}_c^{-1}(j\omega)) > 1 \tag{12}$$

(where $\sigma_{min}(\underline{A})$ denotes the smallest singular value of $\underline{A}$), and

then, provided $L_{yv}(j\omega)\ L_{yv}(j\omega)$ is full rank for all ω,

$$\mathcal{T}\ \underline{\text{inside}}\ L_2\text{-Cone}\ (\mathcal{T}_{nom},\ \mathcal{P}_T^{1/2},\ \mathcal{Q}_T^{1/2}) \tag{13}$$

where $\mathcal{T}_{nom}$, $\mathcal{Q}_T$, and $\mathcal{P}_T$ are the (not necessarily causal) L_2-stable LTI operators specified in terms of their bilateral Laplace transform transfer matrices by

$$\underline{T}_{nom}(s) \triangleq \underline{L}_{yu}(s) + \underline{L}_{yv}(s)\Big(\underline{S}_c^T(-s)\underline{S}_c(s) - \underline{L}_{ev}^T(-s)\underline{R}_c^T(-s)\underline{R}_c(s)\underline{L}_{ev}(s)\Big)^{-1} \cdot \underline{L}_{ev}^T(-s)\underline{R}_c^T(-s)\underline{R}_c(s)\underline{L}_{eu}(s) \tag{14}$$

$$\underline{Q}_T(s) \triangleq \Big(\underline{L}_{yv}(s)\Big(\underline{S}_c^T(-s)\underline{S}_c(s) - \underline{L}_{ev}^T(-s)\underline{R}_c^T(-s)\underline{R}_c(s)\underline{L}_{ev}(s)\Big)^{-1}\underline{L}_{yv}^T(-s)\Big)^{-1} \tag{15}$$

$$\underline{P}_T(s) \triangleq \underline{L}_{eu}^T(-s)\underline{R}_c^T(-s)\Big(\underline{I} - \underline{R}_c(s)\underline{L}_{ev}(s)\Big(\underline{S}_c^T(-s)\underline{S}_c(s)\Big)^{-1}\underline{L}_{ev}(-s)\underline{R}_c^T(-s)\Big)^{-1} \cdot \underline{R}_c(s)\underline{L}_{eu}(s). \tag{16}$$

Moreover, the set L_2-Cone $(\mathcal{T}_{nom},\ \mathcal{P}_T^{1/2},\ \mathcal{Q}_T^{1/2})$ is the tightest possible bound on $\mathcal{T}$ in the sense that for every pair of signals $(\underline{u},\ \underline{y}) \in L_2$-Cone $(\mathcal{T}_{nom},\ \mathcal{P}_T^{1/2},\ \mathcal{Q}_T^{1/2})$, there exists at least one 'perturbation' $\delta\mathcal{C}$ satisfying (7) such that $\underline{y} = \mathcal{T}\ \underline{u}$.

<u>Proof</u>. See Appendix .

DISCUSSION

The uncertainty bound provided by Theorem 1, like the uncertainty bound on the plant ignorance $\delta\mathcal{C}$, takes the form of an L_2-conic sector condition. Consequently, in the special case in which the perturbation $\delta\mathcal{C}$ is linear-time-invariant, Parseval's theorem [5, p. 236] implies that the L_2-conic sector bound on the overall system input-output relation provided by Theorem 1 can be expressed using singular values giving the bound on the system's frequency response matrix

$$\sigma_{max}(Q_T^{1/2}(j\omega)(T(j\omega) - T_{nom}(j\omega))\ P_T^{-1/2}(j\omega)) \leq 1\ ,$$

cf. equation (5). In particular for single-input-single-output LTI systems having uncertain LTI components, this bound becomes

$$|T(j\omega) - T_{nom}(j\omega)|^2 \leq \frac{P_T(j\omega)}{Q_T(j\omega)}\ .$$

[7]For nonanticipative systems, L_2-stability is implied by L_{2e}-stability (which is defined in [6]). It can be shown that if the system (1) is nonanticipative and is L_{2e}-stable when $\delta\mathcal{C} \equiv 0$, then (12) is sufficient to guarantee that (1) is L_{2e}-stable for all $\delta\mathcal{C}$ satisfying (7) [6]. Consequently, at least for nonanticipative systems, condition (b) of Theorem 1 is not very restrictive and may be easily verified by checking that $\mathcal{T}$ is L_{2e}-stable.

Theorem 1 gives the tightest possible bound on the overall system input-output relation T only when all available information about the perturbation $\delta \underset{\sim}{C}$ is contained in condition (7). This is an important limitation, since in general this will not be the case. For example, if bounds on system uncertainty are given in the form of (8) then, although condition (7) holds with $\underset{\sim}{R}_c$ and $\underset{\sim}{S}_c$ specified by (9)-(10), condition (7) does not retain the 'structural' information that $\delta \underset{\sim}{C}$ is diagonal (i.e., that $\delta \underset{\sim}{C}$ has no 'cross-couplings' between components). Consequently, in such situations one may in general expect the bound provided by Theorem 1 to be conservative. In contrast, the classical root-locus and frequency-response methods (cf. [3, Ch. 6] and [4]) for generating 'regions of uncertainty' (viz., root loci and frequency response 'templates') to bound system response need not be conservative. Thus, the response bounds provided by Theorem 1 are the most useful in the case of systems for which these classical approaches are impractical or inapplicable. Such systems include multiloop feedback systems, multi-input-multi-output systems, systems with more than a very few uncertain parameters, and systems with nonlinear components.

The problem formulation used in this paper is sufficiently general to accommodate multiloop 'two degree of freedom' (cf. [12, §6.1], [4]) feedback structures with plant parameter ignorance such as depicted in Figure 2. For this configuration, one finds that the transfer function matrix $\underline{L}(s)$ of (11) is given by (suppressing the argument 's')

$$\left[\begin{array}{c|c} \underline{L}_{yu} & \underline{L}_{yv} \\ \hline \underline{L}_{eu} & \underline{L}_{ev} \end{array}\right] = \left[\begin{array}{c|c} \underline{G}_0\, \underline{C}(\underline{I} + \underline{F}\,\underline{C})^{-1}\underline{P} & \underline{G}_0(\underline{I} + \underline{C}\,\underline{F})^{-1} \\ \hline (\underline{I} + \underline{F}\,\underline{C})^{-1}\,\underline{P} & -\underline{F}(\underline{I} + \underline{C}\,\underline{F})^{-1} \end{array}\right]$$

where $\underline{F} = \underline{G}_F + \underline{G}_I\, \underline{G}_C\, \underline{G}_{FC}$ and $P = \underline{G}_{IC}\, \underline{G}_C\, \underline{G}_{PC}$. As one might expect based on the work of Horowitz et al. (e.g., [4], [12]), it can be shown using Theorem 1 that the 'two degree of freedom' structure of Figure 2 permits one to adjust the amount of system uncertainty and the nominal value $\underset{\sim}{T}_{nom}$ of the closed-loop system input-output relation independently to a certain extent -- this is the subject of a forthcoming paper.

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APPENDIX

Proof of Theorem 1

We begin by establishing a useful lemma.

Lemma A1: If $\underline{Q} \in \mathbf{C}^{n\times n}$ is a positive definite hermitian matrix and if $\underline{B} \in \mathbf{C}^{r\times n}$ is a matrix of full rank $r \leq n$, then the matrix $\tilde{\underline{Q}} = (\underline{B}\,\underline{Q}^{-1}\underline{B}^*)^{-1}$ exists and has the following properties:

i) $\underline{Q} - \underline{B}^*\tilde{\underline{Q}}\,\underline{B}$ is positive semi definite;

ii) if $\underline{S}$ is any matrix such that $\underline{Q} - \underline{B}^*\underline{S}\,\underline{B}$ is positive semi definite, then $\tilde{\underline{Q}} - \underline{S}$ is positive semidefinite

iii) $$(\underline{B}^+)^*\underline{Q}\,\underline{B}^+ = (\underline{B}^+)^*\underline{B}^*\tilde{\underline{Q}}\,\underline{B}\,\underline{B}^+ \tag{A1}$$

$$\underline{B} \cong \underline{Q}\ \underline{B}\ \underline{Q} \quad . \tag{A2}$$

Proof: For any positive definite matrix M and any positive integer m, let C_M^m denote the inner product space of complex m-vectors with inner product $\langle \underline{z}_1, \underline{z}_2 \rangle \triangleq \underline{z}_2^* \underline{M}\, \underline{z}_1$. Consider $\underline{B}$ as a linear mapping C_Q^n into C_I^r. Then the adjoint of $\underline{B}$, which we denote as $\underline{B}^a$, is

$$\underline{B}^a = \underline{Q}^{-1} \underline{B}^*$$

(recall $\underline{B}^*$ denotes the complex-conjugate of $\underline{B}^T$) and $\underline{B}^+$ is the pseudo-inverse of $\underline{B}$ [11, pp. 150-165]. Further,

$$\begin{aligned}\tilde{\underline{Q}} &\triangleq (\underline{B}\,\underline{Q}^{-1}\underline{B}^*)^{-1} = ((\underline{B}\,\underline{Q}^{-1}\underline{B}^*)^{-1}\underline{B}\,\underline{Q}^{-1})\underline{Q}(\underline{Q}^{-1}\underline{B}^*(\underline{B}\,\underline{Q}^{-1}\underline{B}^*)^{-1}) \\ &= (\underline{B}^+)^*\underline{Q}\,\underline{B}^+ .\end{aligned} \tag{A3}$$

For any vector $\underline{x} \in C_Q^n$, let $\underline{x}_{\eta(B)}$ and $\underline{x}_{\eta^\perp(B)}$ denote the respective projections of $\underline{x}$ onto the nullspace of $\underline{B}$ and the orthogonal complement of the nullspace of $\underline{B}$. Now, for any $\underline{x} \in C_Q^n$ and any $\underline{y} \in C_I^r$

$$\underline{B}^+ \underline{B}\,\underline{x} = \underline{x}_{\eta^\perp(B)} \quad , \tag{A4}$$

$$\underline{B}\,\underline{B}^+ \underline{y} = \underline{y} \quad . \tag{A5}$$

It follows that for all $\underline{x}$

$$\underline{x}^* (\underline{Q} - \underline{B}^* \tilde{\underline{Q}}\, \underline{B})\, \underline{x} = \underline{x}_{\eta(B)}^* \underline{Q}\, \underline{x}_{\eta(B)} \geq 0$$

which establishes property (i). For any matrix S satisfying $\underline{Q} - \underline{B}^* \underline{S}\, \underline{B} \geq 0$ and any $\underline{y} \in C_I^r$ we have

$$\begin{aligned}\underline{y}^* \underline{S}\, \underline{y} &= \underline{y}^* (\underline{B}\underline{B}^+)^* \underline{S}\, (\underline{B}\,\underline{B}^+)\, \underline{y} \\ &= (\underline{B}^+ \underline{y})^* \underline{B}^* \underline{S}\, \underline{B}\, (\underline{B}^+ \underline{y}) \\ &\leq (\underline{B}^+ \underline{y})^* \underline{Q}\, (\underline{B}^+ \underline{y}) \\ &= \underline{y}^* \tilde{\underline{Q}}\, \underline{y} \quad ,\end{aligned} \tag{A6}$$

which establishes property (ii). The identity of property (iii) follows by direct substitution of (A2) into (A1).

$\square$

We now prove that, under the conditions of Theorem 1, equation (13) holds. We begin by noting the input-output relation $\underline{y} = \underset{\sim}{T}\, \underline{u}$ defined by (1) can be described equivalently by

$$\begin{bmatrix} \underline{y} \\ \underline{e} \end{bmatrix} = \underset{\sim}{L} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} = \begin{bmatrix} (\underset{\sim}{L}_{yu}\, \underline{u} + \underset{\sim}{L}_{yv}\, \underline{v}) \\ (\underset{\sim}{L}_{eu}\, \underline{u} + \underset{\sim}{L}_{ev}\, \underline{v}) \end{bmatrix} \tag{A7}$$

$$\underline{v} \triangleq \delta \underset{\sim}{C}\, \underline{e} \ ; \tag{A8}$$

we take $(\underline{u}, \underline{y}, e, \underline{f}, \underline{v})$ to be a solution of (1) and (A7). For any $\underline{z} \in L_2$ and any L_2-stable operator $\underset{\sim}{A}$, we use the notation $\hat{\underline{z}}$ and $\hat{A}$ to denote the Fourier transform of $\underline{z}$ and frequency response of $\underset{\sim}{A}$ respectively.

Suppose that (7) holds. Then $(\underline{e}, \underline{f}) \in L_2$-Cone $(\underset{\sim}{C}, \underset{\sim}{R}_c, \underset{\sim}{S}_c)$ and hence

$$0 \geq \| \underset{\sim}{S}_c(\underline{f} - \underset{\sim}{C}\,\underline{e})\|^2_{L_2} - \|\underset{\sim}{R}_c\,\underline{e}\|^2_{L_2}$$

$$= \langle \underset{\sim}{S}_c\,\underline{f} - (\underset{\sim}{S}_c\,\underset{\sim}{C} + \underset{\sim}{R}_c)\,\underline{e},\; \underset{\sim}{S}_c\,\underline{f} - (\underset{\sim}{S}_c\,\underset{\sim}{C} - \underset{\sim}{R}_c)\,\underline{e} \rangle_{L_2}$$

$$= \langle (\underset{\sim}{S}_c + \underset{\sim}{R}_c\underset{\sim}{L}_{ev})\underline{v} + \underset{\sim}{R}_c\underset{\sim}{L}_{eu}\underline{u},\; (\underset{\sim}{S}_c - \underset{\sim}{R}_c\underset{\sim}{L}_{ev})v - \underset{\sim}{R}_c\underset{\sim}{L}_{eu}\underline{u} \rangle_{L_2}$$

(By (1) and (A7))

$$= \langle (\hat{S}_c + \hat{R}_c\hat{L}_{ev})\hat{v} + \hat{R}_c\hat{L}_{eu}\,\hat{u},\; (\hat{S}_c - \hat{R}_c\hat{L}_{ev})\,\hat{v} - \hat{R}_c\hat{L}_{eu}\,\hat{u} \rangle_{L_2}$$

$$= \langle (\hat{v} - \hat{T}_v\,\hat{u}),\; \hat{Q}_v\,(\hat{v} - \hat{T}_v\,\hat{u}) \rangle_{L_2} - \langle \hat{u}, \hat{P}_v\,\hat{u} \rangle_{L_2} \qquad \text{(A9)}$$

where $\underset{\sim}{Q}_v$, $\underset{\sim}{Z}$, $\underset{\sim}{\Theta}$, $\underset{\sim}{T}_v$, and $\underset{\sim}{P}_v$ are the (not necessarily causal) L_2-stable operators specified in terms of their bilateral Laplace transform transfer functions by

$$\underline{Q}_v(s) \triangleq \underline{S}_c^T(-s)\,\underline{S}_c(s) - \underline{L}_{ev}^T(-s)\underline{R}_c^T(-s)\underline{R}_c(s)\underline{L}_{ev}(s) \qquad \text{(A10)}$$

$$Z(s) \triangleq L_{ev}^T(-s)\;R_c^T(-s)\;R_c(s)\;L_{eu}(s) \qquad \text{(A11)}$$

$$\Theta(s) \triangleq L_{eu}^T(-s)\;R_c^T(-s)\;R_c(s)\;L_{eu}(s) \qquad \text{(A12)}$$

$$T_v(s) \triangleq Q_v^{-1}(s)\;Z(s) \qquad \text{(A13)}$$

$$P_v(s) \triangleq \Theta(s) + Z^T(-s)\;Q_v^{-1}(s)\;Z(s) = P_T(s) \qquad \text{(A14)}$$

where the latter equality in (A14) follows using the matrix identity

$$I + A\,(B - C\,A)^{-1}C = (I - A\,B^{-1}C)^{-1}. \qquad \text{(A15)}$$

Thus, applying part (i) of Lemma A1, we have from (A9) that

$$0 \geq \langle \hat{L}_{yv}\,(\hat{v} - \hat{T}_v\,\hat{u}),\; (\hat{L}_{yv}\,\hat{Q}_v^{-1}\,\hat{L}_{yv}^*)^{-1}\,\hat{L}_{yv}\,(\hat{v} - \hat{T}_v\,\hat{u}) \rangle_{L_2} - \langle \hat{u}, \hat{P}_T\,\hat{u} \rangle_{L_2}$$

$$= \langle \hat{y} - \hat{T}_{nom}\,\hat{u},\; \hat{Q}_T\,(\hat{y} - \hat{T}_{nom}\,\hat{u}) \rangle_{L_2} - \langle \hat{u}, \hat{P}_T\,\hat{u} \rangle_{L_2}. \qquad \text{(A16)}$$

Hence, from Parseval's theorem

$$\| \underset{\sim}{Q}_T^{1/2}\,(\underline{y} - \underset{\sim}{T}_{nom}\,\underline{u})\|_{L_2} \leq \|\underset{\sim}{P}_T^{1/2}\,\underline{u}\|_{L_2} \qquad \text{(A17)}$$

from which (13) follows.

specified by

$$\hat{v} = \hat{L}_{yv}^{+} (\hat{y} - \hat{L}_{yu} \hat{u}) + \hat{T}_v \hat{u} \tag{A18}$$

where

$$\hat{L}_{yv}^{+} \triangleq \hat{Q}_v^{-1} \hat{L}_{yv}^{*} (\hat{L}_{yv} \hat{Q}_v^{-1} \hat{L}_{yv}^{*})^{-1} . \tag{A19}$$

Then, (u, y, v) are consistant with (1) and (A7) and (using part (iii) of Lemma (A1) and Parseval's theorem) it follows that for all $(\underline{u}, \underline{y}) \in L_2$-Cone $(\mathcal{T}_{nom}, \mathcal{P}_T^{1/2}, \mathcal{Q}_T^{1/2})$

$$\begin{aligned} 0 &\geq \| \hat{Q}_T^{1/2} (\hat{y} - \hat{T}_{nom} \hat{u}) \|_{L_2}^2 - \| \hat{P}_T^{1/2} \hat{u} \|_{L_2} \\ &= < (\hat{v} - \hat{T}_v \hat{u}), \hat{L}_{yv}^{*} (\hat{L}_{yv} \hat{Q}_v^{-1} \hat{L}_{yv}^{*})^{-1} \hat{L}_{yv} (\hat{v} - \hat{T}_v \hat{u}) >_{L_2} \\ &\qquad - < \hat{u}, \hat{P}_T \hat{u} >_{L_2} \\ &= \| \underset{\sim}{S}_c (\underline{f} - \underset{\sim}{C} \underline{e}) \|_{L_2}^2 - \| \underset{\sim}{R}_c \underline{e} \|_{L_2}^2 \end{aligned} \tag{A20}$$

where the latter equality follows as in (A9). Let $\delta \underset{\sim}{C}$ be the map defined by

$$\delta \underset{\sim}{C} \underline{z} = \| \underset{\sim}{R}_c \underline{e} \|_{L_2}^{-2} < \underset{\sim}{R}_c \underline{e}, \underset{\sim}{R}_c \underline{z} >_{L_2} \underline{v}. \tag{A21}$$

Using (A20) and the Schwartz inequality, one readily verifies that $\delta \underset{\sim}{C} \underline{e} = \underline{v}$ and that $\delta \underset{\sim}{C}$ satisfies (7).

(End of proof)

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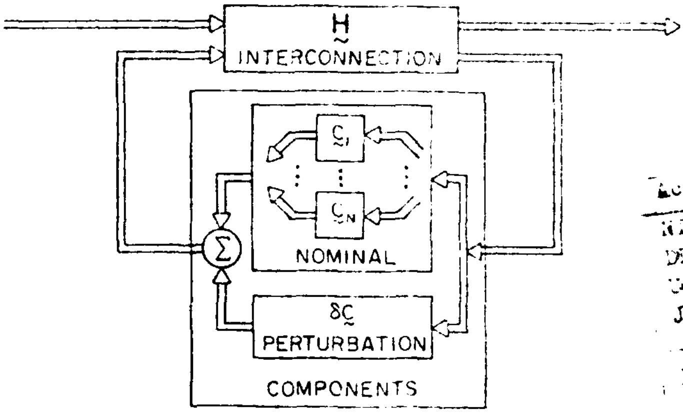

Figure 1. The system.

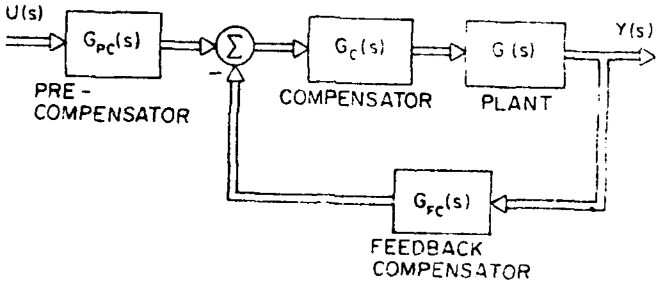

Figure_2(a). Multiloop 'two degree of freedom' feedback control system.

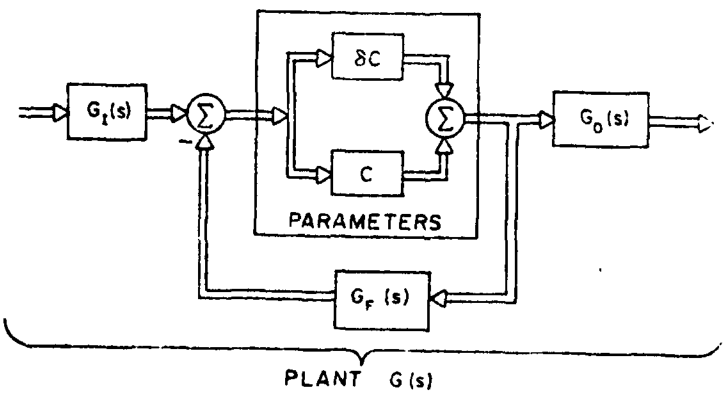

Figure 2(b). Internal structure of uncertain plant in Figure 2(a).

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